STA203 Sunday

Week 2

- part 1 Basic set Theory
- part 2 probability space $(\Omega, \mathcal{A}, P)$
- part 3 Some Combinatorics

σ-algebra / σ-field

$P: \mathcal{A} \to [0,1]$

state space

Question

Give a rigorous proof of the De Morgan's Law

For any index set I e.g. $I = \{1,2,\cdots,m\}$, $I = \{1,2,\cdots,m,\cdots\}$

$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} (A_i)^c$$

$\boxed{A = B}$

① $A \subseteq B$ ② $B \subseteq A$

proof: $\forall x \in LHS = \left(\bigcup_{i\in I} A_i\right)^c$

then $x \notin \left(\bigcup_{i\in I} A_i\right)$

in other words

$\underbrace{x \notin A_i, \forall i \in I}$

$x \in A_i^c, \forall i \in I$

then $x \in \bigcap_{i\in I} A_i^c = RHS$

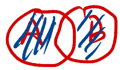
Then $LHS \subseteq RHS$

Similarly $RHS \subseteq LHS$, then done $\square$

Notations

- $A \cup B$
- $A \cap B$
- A^c
- $A \setminus B$ difference. $x \in A.\ x \notin B$
- $A \Delta B$

symmetric difference



$x \in A \cup B.\ x \notin A \cap B$

part 2 : Probability Space

A special measurable space.

$(\Omega, \mathcal{A}, P)$

σ-field

Def σ-field (σ-algebra)

- $\Omega \in \mathcal{A}$
- If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$
- If $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

countably infinity

- If $\{A_i\}_{i=1}^{m} \subseteq \mathcal{A}$, $\bigcup_{i=1}^{m} A_i \in \mathcal{A}$

$\Downarrow$

Algebra

Notation: σ, δ, 2 Greek letters

$\sigma \to$ "Sum" $\to$ union

$\delta \to$ "intersection"

Examples of σ-algebra

- Borel σ-algebra of $\mathbb{R}$, as $\mathcal{B}$

  the smallest σ-algebra containing all open set $\Updownarrow$ closed

- Finite σ-algebra

  $\mathcal{A} = \{\phi, \Omega\}$ trivial case

  $\boxed{\Omega = \{1, 2, 3\}}$

  $\mathcal{A} = \{\phi, \Omega, \{1\}, \{2,3\}\}$

  $\{1\} \to \{2,3\}$, $\{2,3\} \to \{1\}$

  $\{1\} \cup \Omega = \Omega$

  $\{1\} \cup \{2,3\} = \Omega$

  $\phi \cup \{2,3\} = \{2,3\}$

  ...

$\mathcal{A}$ $\quad P : \mathcal{A} \to [0,1]$

P ↓ probability

- $P(\Omega) = 1$
- (σ-additivity)

  For $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ pairwise disjoint.

  $$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

$\underbrace{(\Omega, \mathcal{A}, P)}_{\text{A probability space}}$

Question

3 professors $\quad$ A: Mary $\quad\quad$ A: Mary is awarded

$\quad$ B: Peter

An Award $\to$ A, B, C

$\quad$ C: David

(a) only Mary $\quad A \cap B^c \cap C^c$

(b) at least one of the three $\quad A \cup B \cup C$

(c) exactly two of them $\quad (A \cap B \cap C^c) \cup (A \cap B^c \cap C)$

$\quad \cup (A^c \cap B \cap C)$

(d) Mary or Peter but not both

$$(A \cup B) \setminus (A \cap B)$$

$$= (A \cap B^c) \cup (B \cap A^c)$$

$P: \mathcal{A} \to [0,1]$

Question

28% HK Males Cigarettes $A \Rightarrow P(A) = 0.28$

8% ~ cigars $B \Rightarrow P(B) = 0.08$

6% ~ cigarettes & cigars $A \cap B \Rightarrow P(A \cap B) = 0.06$

- Probability of HK males smokes neither cigars nor cigarettes

$$P((A \cup B)^c) = P(\Omega \setminus (A \cup B))$$
$$= P(\Omega) - P(A) - P(B) + P(A \cap B) = 0.7$$

- probability of ~ smokes cigars but not cigarette

$$P(B \setminus A) = P(B) - P(A \cap B) = \underline{\underline{0.02}}$$

Question

prove that

$$\boxed{P(A \cap B) \geq \underbrace{P(A) + P(B) - 1}_{\text{lower bd.}}}$$

$$\underset{1}{\overset{P(\Omega)}{\|}} \geq \underbrace{P(A) + P(B) - P(A \cap B)}_{P(A \cup B)}$$

$A \cup B \subseteq \Omega$

$P(A \cup B) \leq P(\Omega) = 1$

$1 \geq P(A) + P(B) - P(A \cap B)$

证 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P(A \cap B) = \cancel{P(A \cup B) - P(A)}$.

$\| \qquad P(A) + P(B) - P(A \cup B)$

LHS

原式 $\Leftrightarrow \cancel{P(A)} + \cancel{P(B)} - P(A \cup B) \geq \cancel{P(A)} + \cancel{P(B)} - 1$

$\Leftrightarrow P(A \cup B) \leq 1$.

Question

prove the following

$$\boxed{P(\cap_{i=1}^n A_i) \ge \sum_{i=1}^n P(A_i) - (n-1)}$$

$\Leftrightarrow \quad n - \sum_{i=1}^n P(A_i) \ge 1 - P(\cap_{i=1}^n A_i) \quad = P((\cap_{i=1}^n A_i)^c)$

$= P(\cup_{i=1}^n A_i^c)$

$\Leftrightarrow \quad \sum_{i=1}^n P(\bar{A_i}) \ge 1 - P(\cap_{i=1}^n A_i)$

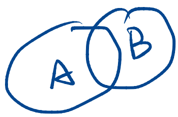
$\sum_{i=1}^n (1-P(A_i)) = \sum_{i=1}^n P(A_i^c)$

$\sum_{i=1}^n P(\bar{A_i}) + P(\cap_{i=1}^n A_i) \ge \qquad + P(\cap_{i=1}^n A_i)$

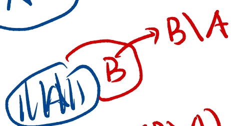
$= 1$

$\Leftrightarrow \quad P(\cup_{i=1}^n A_i^c) \le \sum_{i=1}^n P(A_i^c)$

sub-additivity 次可加



① 

② 

$A_1 \quad A_2 \setminus A_1 \quad A_3 \setminus \cup_{i=1}^2 A_i$

$A_4 \setminus \cup_{i=1}^3 A_i$

$A \cap (B \setminus A) = \phi$

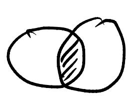
$P(A \cup B) = P(A) + P(B \setminus A)$

$\le P(A) + P(B)$

Question $\quad P(A) = \frac{3}{4} \quad P(B) = \frac{1}{3}$, show that

$$\frac{1}{12} \le P(A \cap B) \le \frac{1}{3}$$

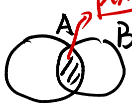
$\frac{1}{12} = \frac{3}{4} + \frac{1}{3} - 1$

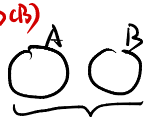


$(A \cap B) \subseteq B$

$P(A \cap B) \le P(B) = \frac{1}{3}$

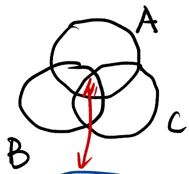
$p(A\cap B)$ $\{p(A), p(B)\}$





$P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$P(A\cup B) = P(A) + P(B)$



$\boxed{A\cap B\cap C}$

$$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(A\cap C) - P(B\cap C) + P(A\cap B\cap C)$$

$+P(A)$, $+P(B)$, $+P(C)$

$-P(A\cap B)$ $-P(A\cap C)$ $-P(B\cap C)$

Thm (PIE. principle of Inclusion and Exclusion)

Let S be a N-set

$\{E_1, \dots, E_r\} \subseteq S$, they are not necessarily distinct

$E_1 \subseteq S$, $E_r \subseteq S$

For any subset $M \subseteq \{1, \dots, r\}$

$$N(M) \overset{def}{=} \#\{s\in S, s\in \cap_{i\in M} E_i\}$$

Also define

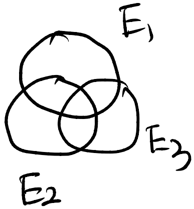
$$N_j \overset{def}{=} \sum_{|M|=j} N(M)$$

Example

$N_2 = \sum_{|M|=2} N(M)$

$P(A\cap B)$ $P(A\cap C)$ $P(B\cap C)$

$$|S\setminus(E_1\cup\cdots\cup E_r)| = N - N_1 + N_2 - N_3 + \cdots + (-1)^r N_r \quad (1)$$



$|S \setminus (E_1 \cup E_2 \cup E_3)|$

<u>proof</u>: • $\forall x \in S$

x are not in any of the r-sets

i.e. $x \in \underline{S \setminus E_1 \cup E_2 \cup E_3}$

then x contribute to (1) by 1

x has no contribution to

N_1 N_2 N_3

• If $x \in S$

is in k many E_i's, then

it contribute to (1)

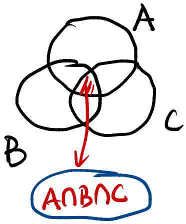
$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} = (1-1)^k = 0.$$

$\square$

<u>Ref</u> A course in Combinatorics

by J.H. van Lint

and R.M. Wilson



$P(A \cup B \cup C)$

$= P(A) + P(B) + P(C)$

$- P(A\cap B) - P(A\cap C) - P(B \cap C)$

$+ P(A \cap B \cap C)$

$+ \cancel{P(A)}, + \cancel{P(B)}, + \cancel{P(C)}$

$- \cancel{P(A\cap B)} - \cancel{P(A\cap C)} - \cancel{P(B\cap C)}$

$\Downarrow$

Q (Bonferroni's Inequality)

$A = A_1 \cup \cdots \cup A_n$

- $P(A) \le \sum_{k=1}^{m} (-1)^{k-1} S_k$, for any odd m

$m=1$

$P(A \cup B \cup C) \le P(A) + P(B) + P(C)$

- $P(A) \ge \sum_{k=1}^{m} (-1)^{k-1} S_k$, for any even m

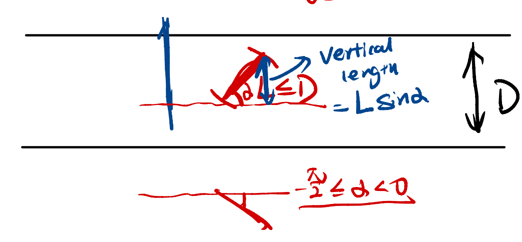
$m=2$

$P(A \cup B \cup C) \ge P(A) + P(B) + P(C) - P(A\cap B) - P(A \cap C) - P(B \cap C)$

where $S_k \overset{\text{def}}{=} \sum_{1 \le i_1 < \cdots < i_k \le n}^{n} P(A_{i_1} \cap \cdots \cap A_{i_k})$

Exercise [Buffon's Needle Problem)

$0 \le \alpha \le \frac{\pi}{2}$



P (the needle intersects one of the Lines)

$\|$

when $0 \le \alpha \le \frac{\pi}{2}$

vertical length $= L\sin\alpha$

For fixed α, $P = \frac{L \sin\alpha}{D}$

then $\Pr\left(\text{the needle intersects one of the lines}\right) = \dfrac{\int_0^{\frac{\pi}{2}} \frac{L}{D} \sin\alpha \, d\alpha}{\frac{\pi}{2}}$

$$= \left(\frac{2}{\pi} \frac{L}{D}\right) \underbrace{\left(\int_0^{\frac{\pi}{2}} \sin\alpha \, d\alpha\right)}_{= 1}$$

$$= \frac{2}{\pi} \frac{L}{D}$$

Needle problem $\Rightarrow$ Approximate π