Modified Version from . by Zhaozhe Liu.

STA203 Foundation of Probability Theory

Week 3

Shengqi FANG Tuesday 24th September, 2024

Based on Probability Theory's slides

- Conditional Probability

Definition 1.1 (Conditional probability). Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and $A, B \in \mathscr{F}$ be two events such that $\mathbb{P}(B) > 0$. The conditional probability of A given B is defined by

$$\mathbb{P}(A|B) = \frac{P(E \cap F)}{P(F)} \qquad \frac{P(A\cap B)}{P(B)}$$

comes from frequency explanation

$$f_n(A|B) = \frac{n f_n(A\cap B)}{n f_n(B)} = \frac{f_n(A\cap B)}{f_n(B)} \quad \text{then take limit}$$

- A, B are independent, if $P(A|B) = P(A)$ (not symmetric)

$$\Updownarrow$$

$$P(A\cap B) = P(A)P(B) \quad \text{(symmetric form)}$$

- $\{A, B, C\}$ are independent (mutually independent)

or $P(B|A) = P(B)$

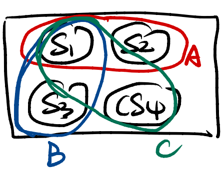
$$\left.\begin{aligned} &A \perp\!\!\!\perp B \cdot P(A|B) = P(A) \Rightarrow \underline{P(A\cap B) = P(A)P(B)} \\ &B \perp\!\!\!\perp C \cdot P(C|B) = P(C) \Rightarrow P(B\cap C) = P(B)P(C) \\ &C \perp\!\!\!\perp A \cdot P(A|C) = P(A) \Rightarrow P(A\cap C) = P(A)P(C) \end{aligned}\right\} \text{pair-wise independent}$$

- $P(A\cap B\cap C) = P(A)P(B)P(C)$

Counter-example [pairwise but not mutual Independence]

$$pr(A) = \tfrac{1}{2} = P(B) = P(C)$$

$$S = \{S_1, S_2, S_3, S_4\} \quad pr(S_i) = \tfrac{1}{4}$$



$$A = \{S_1, S_2\}$$
$$B = \{S_1, S_3\}$$
$$C = \{S_1, S_4\}$$

$$pr(A\cap B) = pr(\{S_1\}) = \tfrac{1}{4}$$
$$pr(A\cap C) = pr(\{S_1\}) = \tfrac{1}{4}$$
$$pr(B\cap C) = pr(\{S_1\}) = \tfrac{1}{4}$$
$$pr(A\cap B\cap C) = pr(\{S_1\}) = \tfrac{1}{4} \neq P(A)P(B)P(C)$$

Def : Suppose $\{A_i\}_{i\in I}$ the family is independent

if for any finite subset J of I $(J \subseteq I, |J| < \infty)$

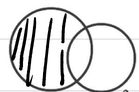
$$P(\cap_{i\in J} A_i) = \Pi_{i\in J} P(A_i)$$

Theorem: If A and B are independent so are A and B^c, A^c and B, A^c and B^c

Already taught at class could leave it as an exercise again

Proof: ① $P(AB^c) = P(A\setminus AB)$

$= P(A) - P(AB) = P(A) - P(A)P(B) = P(A)P(B^c)$



A B

the others are of the same principle.

$$P(A\cap B^c) = P(A\setminus A\cap B)$$

$$= P(A) - P(A\cap B)$$

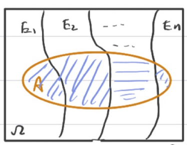
$$= P(A) - P(A)P(B) = P(A)P(B^c)$$

$A, B \Rightarrow A\cap B = \emptyset$

Definition: A sequence of events $\{E_n\}_{n\geq 1}$ is a partition of Ω if it is pair-wise disjoint and the union is Ω.

set-theory concept
$\Downarrow$
mutually exclusive (probability)

Note that $A \perp B$, independent
$A\cap B \neq \emptyset$, not exclusive !



E_1 E_2 --- E_n A Ω

Theorem (Bayes formula)

$$P(E_n|A) = \frac{P(A|E_n)P(E_n)}{\sum_m P(A|E_m)P(E_m)}$$

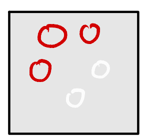
Proof: $P(E_n|A) = \frac{P(AE_n)}{P(A)} = \frac{P(A|E_n)P(E_n)}{P(A)} \longrightarrow \sum_m P(AE_m) = \sum_m P(A|E_m)P(E_m)$ ▨

一般是从复杂的一边⇒简单的一边

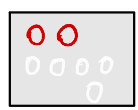
Exercise 1.2. Urn A has three red balls and two white balls, and urn B has two red balls and five white balls. A fair coin is tossed; if it hands heads up, a ball is drawn from urn A and otherwise a ball is drawn from urn B.

- (a)What is the probability that a red ball is drawn ?
- (b) If a red ball is drawn, what is the probability that the coin landed heads up?

Solution.
- (a)$\mathbb{P}(red) = \frac{3}{5}\times\frac{1}{2} + \frac{2}{7}\times\frac{1}{2} = \frac{31}{70}$ ✓
- (b)$\mathbb{P}(head|red) = \frac{21}{31}$ □



A $\# = 5$



B $\# = 7$

(a) $P(red) = P(red \mid A\cup B)$

$$= \frac{P(red\cap A) + P(red\cap B)}{P(A\cup B)}$$

$$= P(red|A)P(A) + P(red|B)P(B)$$

$$= \frac{3}{5}\times\frac{1}{2} + \frac{2}{7}\times\frac{1}{2}$$

$P(A|red)$

$$= \frac{P(A\cap red)}{P(red)} = \frac{\frac{1}{2}\times\frac{3}{5}}{\frac{31}{70}}$$

$$= \frac{21}{31}$$

Exercise

Sometimes the conditional probability is easier to compute, we want to find probability for intersection event. $p(AB) = p(A|B)\cdot p(B)$

$\Downarrow$ 推广到 n 个

Theorem

$$P(A_1A_2\cdots A_n) = P(A_1)P(A_2|A_1)\cdots P(A_n|A_1\cdots A_{n-1}).$$

proof: By induction. Suppose $n=2$, It holds because $p(A_2|A_1) = \dfrac{p(A_1A_2)}{p(A_1)}$

Suppose it holds for $n-1$, now for n.

$$p(\underbrace{A_1A_2\cdots A_{n-1}}_{B}A_n) = p(A_1A_2\cdots A_{n-1})\,p(A_n|A_1\cdots A_{n-1})$$

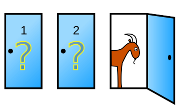
$$= p(A_1)p(A_2|A_1)\cdots p(A_{n-1}|A_1\cdots A_{n-2})\,p(A_n|A_1\cdots A_{n-1}) \quad \blacksquare$$

Exercise 1.5. Show that if A; B; C are mutually independent, then $A\cap B$ and C are independent and $A\cup B$ and C are independent.

Proof. Omit the proof.

$$p[(A\cap B)\cap C] = p(A)p(B)p(C) = [p(A\cap B)]p(C)$$

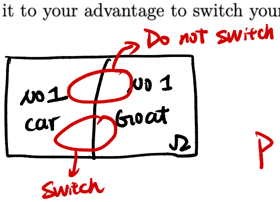
$$p[(A\cup B)\cap C] = p[(A\cap C)\cup(B\cap C)]$$
$$= p(A\cap C) + p(B\cap C) - p(A\cap B\cap C)$$
$$= p(A)p(C) + p(B)p(C) - p(A)p(B)p(C)$$
$$= p(C)\,[\underbrace{p(A)+p(B) - p(A)p(B)}_{p(A\cup B)}]$$



Exercise 1.9 (Monty Hall Problem). Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

$$\begin{cases} p(\text{No 1-car}) = \frac{1}{3} \\ p(\text{No 1-Goat}) = \frac{2}{3} \end{cases}$$

$p(\text{No 1-car} \mid \text{No3-Goat}) = \frac{1}{2}$



$$p(\text{Do NOT SWITCH}) = p(D \mid \text{No 1-car})\,p(\text{No1-car}) + p(D \mid \text{No1-Goat})\,p(\text{No1-Goat})$$
$$= 1\times\frac{1}{3} + 0\times\frac{2}{3} = \frac{1}{3}$$

$$p(\text{SWITCH}) = p(S \mid \text{No 1-car})\,p(\text{No 1-car}) + p(S \mid \text{No1-Goat})\,p(\text{No 1-Goat})$$
$$= 0\times\frac{1}{3} + 1\times\frac{2}{3} = \frac{2}{3}$$

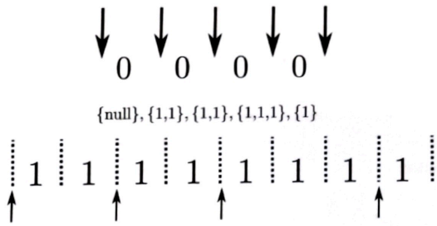
Supplement Exercise (stars and bars)

▼ (d)

How many sequences $(a_1, a_2, \dots, a_{12})$ are there consisting of 4 0's and 8 1's, if no two consecutive terms are both 0's?

Solution:

There are 5 places ready for insertion of 1, The beginning and ending place can be inserted by null



then There are $\binom{9}{4} = 126$ possible sequences with the desired properties. ✓

Some properties of Probability Measure

Example 2

Prove that $\bigcup_{n=1}^{\infty}(0, \frac{n}{n+1}] = (0,1)$.

Note that $\frac{n}{n+1} = 1 - \frac{1}{n+1} < 1$

In fact

$\forall (a, b)$

$= \bigcup_{n=n_0}^{\infty} (a, b - \frac{1}{n_0}]$

Thm Any open set of $\mathbb{R}$ U. U is a countable union of disjoint open intervals

Proof.

(i) Step 1: $\bigcup_{n=1}^{\infty}(0, \frac{n}{n+1}] \subset (0,1)$. Let $x \in \bigcup_{n=1}^{\infty}(0, \frac{n}{n+1}]$, then $x \in (0, \frac{n}{n+1}]$ for some $n \geqslant 1$. Thus, $0 \leqslant x \leqslant \frac{n}{n+1} < 1$, which implies that $x \in (0,1)$. Easy

(ii) Step 2: $(0,1) \subset \bigcup_{n=1}^{\infty}(0, \frac{n}{n+1}]$. Let $x \in (0,1)$, and define $\varepsilon = 1 - x > 0$. Then, there exists a number N such that

$$\varepsilon > \left|\frac{N}{N+1} - 1\right|.$$

$\frac{N}{N+1} > x$

$1 - \frac{1}{N+1} > x$

$1 - x > \frac{1}{N+1}$

$\varepsilon > \frac{1}{N+1}$

Therefore,

$$1 - x = \varepsilon > 1 - \frac{N}{N+1} \implies x < \frac{N}{N+1}.$$

Hence,

$$x \in (0, \frac{N}{N+1}] \in \bigcup_{n=1}^{N}(0, \frac{n}{n+1}] \in \bigcup_{n=1}^{\infty}(0, \frac{n}{n+1}].$$

If $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$, $A = \bigcup_{i=1}^{\infty} A_i$, we write $A_i \uparrow A$

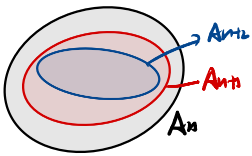
$A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$ $A = \bigcap_{i=1}^{\infty} A_i$ $A_i \downarrow A$

Theorem: $\mathcal{A}$ be a σ-algebra, Suppose $P(\Omega)=1$, and is additive (finite). then the followings are equivalent:

1. P is σ-additive.
2. If $A_n \in \mathcal{A}$, and $A_n \downarrow \phi$, then $P(A_n) \downarrow 0$ $\Rightarrow$ 常用
3. $A_n \in \mathcal{A}$, $A_n \downarrow A$, then $P(A_n) \downarrow P(A)$ [Continuity from above]
4. ----, $A_n \uparrow \Omega$, then $P(A_n) \uparrow 1$.
5. ---, $A_n \uparrow A$, then $P(A_n) \uparrow P(A)$ [Continuity from below]

proof: $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$

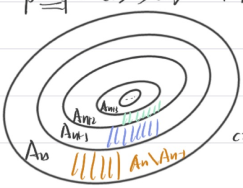
$(1) \Rightarrow (2)$: $\underbrace{A_n \setminus A_{n+1},\ A_{n+1} \setminus A_{n+2}}_{\text{disjoint}}$



and $A_n = \bigcup_{k=n}^{\infty} (A_k \setminus A_{k+1})$

$$P(A_n) = \sum_{k=n}^{\infty} P(A_k \setminus A_{k+1}) \downarrow 0, \quad n \to \infty$$

proof: $(1) \Rightarrow (2)$, As $A_n \setminus A_{n+1}$, $A_{n+1} \setminus A_{n+2}$, $A_{n+2} \setminus A_{n+3}$, ---, are disjoint.



and $\bigcup_{k=n}^{\infty} (A_k \setminus A_{k+1}) = A_n$, 无穷级数的余项

we have $P(A_n) = P(\bigcup_{k=n}^{\infty} (A_k \setminus A_{k+1})) = \sum_{k=n}^{\infty} P(A_k \setminus A_{k+1}) \downarrow$, as $n \to \infty$

$(2) \Rightarrow (3)$, As $A_n \setminus A \downarrow \phi$, then $P(A_n \setminus A) \downarrow 0$

$\because A_n = (A_n \setminus A) \cup A$, then $P(A_n) = P(A_n \setminus A) + P(A) \downarrow P(A)$ (finite-add)

$(3) \Rightarrow (4)$ As $A_n \uparrow \Omega$, then, $A_n^c \downarrow \phi$, $P(A_n^c) \downarrow 0$. $\because P(A_n^c) = 1 - P(A_n) \downarrow 0$

then $P(A_n) \uparrow 1$.

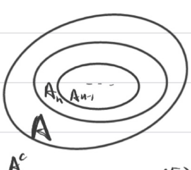
$(3) \Rightarrow (4)$ As $A_n \uparrow \Omega$, then, $A_n^c \downarrow \phi$, $P(A_n^c) \downarrow 0$. $\because P(A_n^c) = 1 - P(A_n) \downarrow 0$

then $P(A_n) \uparrow 1$.

$(4) \Rightarrow (5)$. As $(A_n \cup A^c) \uparrow \Omega$, then $P(A_n \cup A^c) \uparrow 1$

as $A_n \cap A^c = \phi$. then $P(A_n \cup A^c) = P(A_n) + P(A^c) = P(A_n) + 1 - P(A) \uparrow 1$

$\therefore P(A_n) \uparrow P(A)$.



$(5) \Rightarrow (1)$. let A_n be pair-wise disjoint want to prove $P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$

LHS $= \lim_{N \to \infty} P(\bigcup_{n=1}^{N} A_n)$, as $\bigcup_{n=1}^{N} A_n \uparrow \bigcup_{n=1}^{\infty} A_n$, then $P(\bigcup_{n=1}^{N} A_n) \uparrow P(\bigcup_{n=1}^{\infty} A_n)$

$= \lim_{N \to \infty} \sum_{n=1}^{N} P(A_n) = \sum_{n=1}^{\infty} P(A_n) = $ RHS ///

最大下界

$\therefore \overline{\lim_{n\to\infty}} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$, $\underline{\lim_{n\to\infty}} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$, $\lim_{n\to\infty} A_n = \overline{\lim_{n\to\infty}} A_n = \underline{\lim_{n\to\infty}} A_n$

最小上界

- If $\{A_n\}$ is monotone, the above definition also holds.

def: $\lim_{n\to\infty} A_n = A$, if $\overline{\lim_{n\to\infty}} A_n = \underline{\lim_{n\to\infty}} A_n = A$.

(要理解)

- Note that $w \in \limsup_{n\to\infty} A_n$, iff $\forall n, \exists m \geq n$, s.t. $w \in A_m$, $w \in A_n$ infinitely often.

  $w \in \liminf_{n\to\infty} A_n$, iff $\exists n, \forall m \geq n$, s.t. $w \in A_m$

  $\Downarrow$ $w \in A_n$ all but finite many.

  (要掌握)

  $\Downarrow$

  Remark: $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$ (由定义)

- $(\liminf_{n\to\infty} A_n)^c = (\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k)^c = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k^c = \limsup_{n\to\infty} A_n^c$

  取余集.